

TRANSVERSAL INTERSECTION CURVES OF HYPERSURFACES IN \mathbb{R}^5

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Abstract

In this paper we present the algorithms for calculating the differential geometric properties $\{t, n, b_1, b_2, b_3, \kappa_1, \kappa_2, \kappa_3, \kappa_4\}$, geodesic curvature and geodesic torsion of the transversal intersection curve of four hypersurfaces (given by parametric representation) in Euclidean space \mathbb{R}^5 . In transversal intersection the normals of the surfaces at the intersection point are linearly independent, while as in nontransversal intersection the normals of the surfaces at the intersection point are linearly dependent.

Keywords: Hypersurfaces, transversal intersection, non-transversal intersection.

1. Introduction

The surface-surface intersection problem is a fundamental process needed in modeling shapes in CAD/CAM system. It is useful in the representation of the design of complex objects and animations. The two types of surfaces most used in geometric designing are parametric and implicit surfaces. For that reason different methods have been given for either parametric-parametric or implicit-implicit surface intersection curves in \mathbb{R}^3 . The numerical marching method is the most widely used method for computing the intersection curves in \mathbb{R}^3 and \mathbb{R}^4 . The marching method involves generation of sequences of points of an intersection curve in the direction prescribed by the local geometry (Bajaj et al., 1988; Patrikalakis, 1993). To compute the intersection curve with precision and efficiency, approaches of superior order are necessary, that is, they are needed to obtain the geometric properties of the intersection curves. While differential geometry of a parametric curve in \mathbb{R}^3 can be found in textbooks such as Struik (1950), Willmore (1959), Stoker (1969), Spivak (1975), do Carmo (1976), differential geometry of a parametric curve in \mathbb{R}^n can be found in the textbook such as in klingenber (1978) and in the contemporary literature on Geometric Modelling (Farin, 2002; Hoschek and

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Lasser 1993), but there is only a scarce of literature on the differential geometry of intersection curves. Willmore (1959) and Aléssio (2006) presented algorithms to obtain the unit tangent, unit principle normal, unit binormal, curvature and torsion of the transversal intersection curve of two implicit surfaces. Hartmann (1996) provided formulas for computing the curvature of the intersection curves for all types of intersection problems in \mathbb{R}^3 . Ye and Maekawa (1999) presented algorithms for computing the differential geometric properties of both transversal and tangential intersection curves of two surfaces. Aléssio (2009) formulated the algorithms for obtaining the geometric properties of intersection curves of three implicit hypersurfaces in \mathbb{R}^4 . Based on the work of Aléssio (2009), Mustafa Düldül (2010) worked with three parametric hypersurfaces in \mathbb{R}^4 to derive the algorithms for differential geometric properties of transversal intersection. Abdel-All et al. (2012) provided algorithms for geometric properties of implicit-implicit-parametric and implicit-parametric-parametric hypersurfaces in \mathbb{R}^4 . Naeim-Badr et al. (2014) obtained algorithms for differential geometric properties of non-transversal intersection curves of three parametric hypersurfaces in \mathbb{R}^4 . Recently Naeim Badr, Abdel-All et al. (2015) derived the algorithms for non-transversal intersection curves of implicit-parametric-parametric and implicit-implicit-parametric hypersurfaces in \mathbb{R}^4 . To obtain the first geodesic curvature ($\kappa_{1g}^{S_i}$) and the first geodesic torsion ($\tau_{1g}^{S_i}$) for the transversal intersection curve of 4 parametric hypersurfaces in \mathbb{R}^5 , we need to derive the Darboux frame $\{U_1^{M_i}, \dots, U_5^{M_i}\}$. The Darboux frame is obtained by using the Gram-Schmidt orthogonalization process.

In this paper we extended the methods of Mustafa Düldül[8], to obtain the Frenet frame $\{t, n, b_1, b_2, b_3\}$ and curvatures $\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ of transversal intersection curve of four parametric hypersurfaces in \mathbb{R}^5 . In section 2 we introduce some notations and reviews of the differential geometry of curves and surfaces in \mathbb{R}^5 . In section 3 we find the formulas for computing the properties of transversal intersection of four parametric hypersurfaces in \mathbb{R}^5 . In section 4 we derive the formulas for obtaining the geodesic curvature and geodesic torsion of the intersecting curve with respect to four hypersurfaces. Finally, to be more constructive we present an example in section 5. Moreover in addition to the use of classical results of differential geometry we will also make use of Matlab/Mathematica.

2. Preliminaries

Definition 2.1. Let e_1, e_2, e_3, e_4, e_5 be the standard basis of five dimensional Euclidean space E^5 . The vector product of the vectors $x = \sum_{i=1}^5 x_i e_i$, $y = \sum_{i=1}^5 y_i e_i$, $z = \sum_{i=1}^5 z_i e_i$ and $w = \sum_{i=1}^5 w_i e_i$ is defined by

$$x \otimes y \otimes z \otimes w = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{vmatrix} \quad (1)$$

The vector product $x \otimes y \otimes z \otimes w$ yields a vector that is orthogonal to x, y, z, w .

let $R \subset E^5$ be a regular hypersurface given by $\Phi = \Phi(u_1, u_2, u_3, u_4)$ and $\alpha : I \subset \mathbb{R} \rightarrow \Phi$ be an arbitrary curve with arc length parametrisation. If t, n, b_1, b_2, b_3 is the Frenet Frame along α

$$\begin{aligned} t' &= \kappa_1 n \\ n' &= -\kappa_1 t + \kappa_2 b_1 \\ b_1' &= -\kappa_2 n + \kappa_3 b_2 \\ b_2' &= -\kappa_3 b_1 + \kappa_4 b_3 \\ b_3' &= -\kappa_4 b_2 \end{aligned} \quad (2)$$

Where t, n, b_1, b_2 and b_3 denote the tangent, the principal normal, the first binormal, the second binormal and third binormal vector fields. The normal vector n is the normalised acceleration vector α'' . The unit vector b_1 is determined such that n' can be decomposed into two components, a tangent one in the direction of t and a normal one in the direction of b_1 . The unit vector b_2 is determined such that b_1' can be decomposed into two components, a normal one and b_2 . The unit vector b_3 is the unique unit vector field perpendicular to four dimensional subspace $\{t, n, b_1, b_2\}$. The functions $\kappa_1, \kappa_2, \kappa_3$ and κ_4 are the first, second, third and fourth curvatures of $\alpha(s)$. The first, second, third and fourth curvatures measures how rapidly the curve pulls away in a neighbourhood of s , from the tangent line, from planar curve, from three dimensional curve and from the four dimensional curve at s , respectively.

Now, using the Frenet Frame we have the derivatives of α as

$$\alpha' = t, \quad \alpha'' = t' = \kappa_1 n, \quad \alpha''' = -\kappa_1^2 t + \kappa_1' n + \kappa_1 \kappa_2 b_1 \quad (3)$$

$$\alpha^{(4)} = -3\kappa_1 \kappa_1' t + (-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) n + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') b_1 + \kappa_1 \kappa_2 \kappa_3 b_2 \quad (4)$$

$$\begin{aligned} \alpha^{(5)} = & (-3(\kappa_1')^2 - 4\kappa_1 \kappa_1'' + \kappa_1^4 + \kappa_1^2 \kappa_2^2) t + (-6\kappa_1^2 \kappa_1' + \kappa_1''' - \kappa_1' \kappa_2^2 \\ & - 3\kappa_1 \kappa_2 \kappa_2' - 2\kappa_1' \kappa_2^2) n + (\kappa_1^3 \kappa_2 - \kappa_1 \kappa_2^3 + 3\kappa_1' \kappa_2 + 3\kappa_1 \kappa_2' + \kappa_1 \kappa_2'' - \kappa_1 \kappa_2 \kappa_3^2) b_1 \\ & (3\kappa_1' \kappa_2 \kappa_3 + 2\kappa_1 \kappa_2' \kappa_3 + 2\kappa_1 \kappa_2 \kappa_3' + \kappa_1 \kappa_2 \kappa_3') b_2 + \kappa_1 \kappa_2 \kappa_3 \kappa_4 b_3 \end{aligned} \quad (5)$$

Also since Φ is regular, the partial derivatives $\Phi_1, \Phi_2, \Phi_3, \Phi_4$, where $(\Phi_i = \frac{\partial \Phi}{\partial u_i})$ are linearly independent at every point of Φ , i.e., $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4 \neq 0$. Thus, the unit normal vector of Φ is given by

$$N = \frac{\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4}{\|\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4\|}.$$

Furthermore, the first, second, and the third binormal vectors of the curve are given by

$$b_3 = \frac{\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}}{\|\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}\|}, \quad b_2 = \frac{b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''}{\|b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''\|}, \quad b_1 = \frac{b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''}{\|b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''\|} \quad (6)$$

and the curvatures are obtained with

$$\kappa_1 = \|\alpha''\|, \quad \kappa_2 = \frac{\langle \alpha''', b_1 \rangle}{\kappa_1}, \quad \kappa_3 = \frac{\langle \alpha^{(4)}, b_2 \rangle}{\kappa_1 \kappa_2}, \quad \kappa_4 = \frac{\langle \alpha^{(5)}, b_3 \rangle}{\kappa_1 \kappa_2 \kappa_3} \quad (7)$$

On the other hand, since the curve $\alpha(s)$ lies on Φ , we may write

$$\alpha(s) = \Phi(u_1(s), u_2(s), u_3(s), u_4(s)).$$

We then have

$$\alpha'(s) = \sum_{i=1}^4 \Phi_i u'_i \quad (8)$$

$$\alpha''(s) = \sum_{i=1}^4 \Phi_i u''_i + \sum_{i,j=1}^4 \Phi_{ij} u'_i u'_j \quad (9)$$

$$\alpha'''(s) = \sum_{i=1}^4 \Phi_i u'''_i + 3 \sum_{i,j=1}^4 \Phi_{ij} u''_i u'_j + \sum_{i,j,k=1}^4 \Phi_{ijk} u'_i u'_j u'_k \quad (10)$$

$$\begin{aligned} \alpha^{(4)}(s) = & \sum_{i=1}^4 \Phi_i u_i^{(4)} + 4 \sum_{i,j=1}^4 \Phi_{ij} u'''_i u'_j + 3 \sum_{i,j=1}^4 \Phi_{ij} u''_i u''_j + 6 \sum_{i,j,k=1}^4 \Phi_{ijk} u''_i u'_j u'_k + \\ & \sum_{i,j,k,l=1}^4 \Phi_{ijkl} u'_i u'_j u'_k u'_l \end{aligned} \quad (11)$$

$$\begin{aligned} \alpha^{(5)} = & \sum_{i=1}^4 \Phi_i u_i^{(5)} + 5 \sum_{i,j=1}^4 \Phi_{ij} u_i^{(4)} u'_j + 10 \sum_{i,j=1}^4 \Phi_{ij} u'''_i u''_j + 10 \sum_{i,j,k=1}^4 \Phi_{ijk} u'''_i u'_j u'_k \\ & + 15 \sum_{i,j,k=1}^4 \Phi_{ijk} u''_i u''_j u'_k + 10 \sum_{i,j,k,l=1}^4 \Phi_{ijk} u''_i u'_j u'_k u'_l \\ & + \sum_{i,j,k,l,r=1}^4 \Phi_{ijklr} u'_i u'_j u'_k u'_l u'_r \end{aligned} \quad (12)$$

3. The curvature of the transversal intersection of hypersurfaces

Let R_1, R_2, R_3 and R_4 be four regular transversally intersecting hypersurfaces given by parametric equations $\Phi^i = \Phi^i(u_1^i, u_2^i, u_3^i, u_4^i)$, ($i = 1, 2, 3, 4$). Then the unit normal vector of these hypersurfaces are

$$N_i = \frac{\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i}{\|\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i\|}, \quad i = 1, 2, 3, 4$$

Since the intersection is transversal, the normal vectors of these hypersurfaces at the intersection points are linearly independent, i.e., $N_1 \otimes N_2 \otimes N_3 \otimes N_4 \neq 0$. It is assumed that the intersection is a smooth curve say α , in E^5 . Let the intersection curve α be parameterised by arc length function s . Then, at the intersection point $\alpha(s_0) = P$, the unit tangent vector t of the intersection curve α can be found by the vector product of the normal vectors at P .

$$t = \frac{N_1 \otimes N_2 \otimes N_3 \otimes N_4}{\|N_1 \otimes N_2 \otimes N_3 \otimes N_4\|} \quad (13)$$

3.1. First curvature of the transversal intersection curve

Now, we find the first curvature of the intersection curve at P . Since t' is orthogonal to t , we may write

$$\alpha'' = t' = \sum_{i=1}^4 a_i N_i, \quad a_i \in \mathbb{R}, i = 1, 2, 3, 4 \quad (14)$$

Thus, we need to calculate the scalars a_i to find α'' . If we take the dot product of both hand sides of (14) with N_i , we have

$$\begin{aligned} \langle N_1, N_1 \rangle a_1 + \langle N_2, N_1 \rangle a_2 + \langle N_3, N_1 \rangle a_3 + \langle N_4, N_1 \rangle a_4 &= \kappa_n^1, \\ \langle N_1, N_2 \rangle a_1 + \langle N_2, N_2 \rangle a_2 + \langle N_3, N_2 \rangle a_3 + \langle N_4, N_2 \rangle a_4 &= \kappa_n^2, \\ \langle N_1, N_3 \rangle a_1 + \langle N_2, N_3 \rangle a_2 + \langle N_3, N_3 \rangle a_3 + \langle N_4, N_3 \rangle a_4 &= \kappa_n^3, \\ \langle N_1, N_4 \rangle a_1 + \langle N_2, N_4 \rangle a_2 + \langle N_3, N_4 \rangle a_3 + \langle N_4, N_4 \rangle a_4 &= \kappa_n^4 \end{aligned} \quad (15)$$

Where, $\kappa_n^i = \langle t', N_i \rangle$, $(i = 1, 2, 3, 4)$ and $\langle \cdot, \cdot \rangle$ is the scalar product. Hence we must compute $\kappa_n^1, \kappa_n^2, \kappa_n^3, \kappa_n^4$ at P to find the scalars a_i . On using (9), we obtain

$$\begin{aligned} \kappa_n^i &= \Pi_{11}^i (u_1^{i'})^2 + \Pi_{22}^i (u_2^{i'})^2 + \Pi_{33}^i (u_3^{i'})^2 + \Pi_{44}^i (u_4^{i'})^2 + 2(\Pi_{12}^i u_1^{i'} u_2^{i'} + \Pi_{13}^i u_1^{i'} u_3^{i'} \\ &\quad + \Pi_{14}^i u_1^{i'} u_4^{i'} + \Pi_{23}^i u_2^{i'} u_3^{i'} + \Pi_{24}^i u_2^{i'} u_4^{i'} + \Pi_{34}^i u_3^{i'} u_4^{i'}) \end{aligned} \quad (16)$$

Where Π_{lm}^i , $l, m = 1, 2, 3, 4$ are the second fundamental form coefficients of the hypersurfaces Φ^i . Since the unit tangent is known from (13) and $\|\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i\| \neq 0$, the scalar multiplication of both hand sides of (8) with $\Phi_1^i, \Phi_2^i, \Phi_3^i$ and Φ_4^i , respectively yields a linear system of four equations as

$$\begin{aligned} \Upsilon_{11}^i u_1^{i'} + \Upsilon_{12}^i u_2^{i'} + \Upsilon_{13}^i u_3^{i'} + \Upsilon_{14}^i u_4^{i'} &= \langle t, \Phi_1^i \rangle \\ \Upsilon_{21}^i u_1^{i'} + \Upsilon_{22}^i u_2^{i'} + \Upsilon_{23}^i u_3^{i'} + \Upsilon_{24}^i u_4^{i'} &= \langle t, \Phi_2^i \rangle \\ \Upsilon_{31}^i u_1^{i'} + \Upsilon_{32}^i u_2^{i'} + \Upsilon_{33}^i u_3^{i'} + \Upsilon_{34}^i u_4^{i'} &= \langle t, \Phi_3^i \rangle \\ \Upsilon_{41}^i u_1^{i'} + \Upsilon_{42}^i u_2^{i'} + \Upsilon_{43}^i u_3^{i'} + \Upsilon_{44}^i u_4^{i'} &= \langle t, \Phi_4^i \rangle \end{aligned} \quad (17)$$

with respect to $u_1^{i'}, u_2^{i'}, u_3^{i'}$ and $u_4^{i'}$ where Υ_{lm}^i , $l, m = 1, 2, 3, 4$ are the first fundamental form coefficients of the hypersurface Φ^i . Substituting the solutions of these systems into (16) gives us κ_n^i , $i = 1, 2, 3, 4$. Then using matlab/mathematica the system of linear equations in (15) can be solved for a_1, a_2, a_3 and a_4 . Thus the first curvature of the intersection curve at P is obtained from (14) and the first equation of (7).

Remark 3.1. If the normal vectors are mutually orthogonal to each other at the intersection point, then the first curvature is given by $\kappa_1 = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$.

3.2. Second curvature

To compute the second curvature we have to find the third derivative of the intersection curve α .

Since $N_i, i = 1, 2, 3, 4$ are orthogonal to t , we may write $c_1N_1 + c_2N_2 + c_3N_3 + c_4N_4$ instead of terms $\kappa_1' n + \kappa_1 \kappa_2 b_1$ in α''' , i.e.,

$$\alpha''' = -\kappa_1^2 t + c_1N_1 + c_2N_2 + c_3N_3 + c_4N_4. \quad (18)$$

Taking the dot product of both hand sides of above equation with N_i , we obtain

$$\begin{aligned} \langle N_1, N_1 \rangle c_1 + \langle N_2, N_1 \rangle c_2 + \langle N_3, N_1 \rangle c_3 + \langle N_4, N_1 \rangle c_4 &= \mu_1 \\ \langle N_1, N_2 \rangle c_1 + \langle N_2, N_2 \rangle c_2 + \langle N_3, N_2 \rangle c_3 + \langle N_4, N_2 \rangle c_4 &= \mu_2 \\ \langle N_1, N_3 \rangle c_1 + \langle N_2, N_3 \rangle c_2 + \langle N_3, N_3 \rangle c_3 + \langle N_4, N_3 \rangle c_4 &= \mu_3 \\ \langle N_1, N_4 \rangle c_1 + \langle N_2, N_4 \rangle c_2 + \langle N_3, N_4 \rangle c_3 + \langle N_4, N_4 \rangle c_4 &= \mu_4 \end{aligned} \quad (19)$$

Where $\mu_i = \langle \alpha''', N_i \rangle, i = 1, 2, 3, 4$ and $c_i \in \mathbb{R}$. Now, let us find the unknown scalars μ_i . Using (10), we have

$$\mu_r = 3 \sum_{i,j=1}^4 \langle \Phi_{ij}^r, N_r \rangle u_i'' u_j' + \sum_{i,j,k=1}^4 \langle \Phi_{ijk}^r, N_r \rangle u_i' u_j' u_k' \quad (20)$$

Since the components u_i' are known from the system (17), we have to find u_i'' . To obtain u_i'' we use (9) and write

$$\Delta_r = \sum_{i,j=1}^4 \Phi_{ij}^r u_i' u_j', \quad r = 1, 2, 3, 4 \quad (21)$$

Then we have

$$\begin{aligned} \Upsilon_{11}^i u_1^{i''} + \Upsilon_{12}^i u_2^{i''} + \Upsilon_{13}^i u_3^{i''} + \Upsilon_{14}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_1^i \rangle \\ \Upsilon_{21}^i u_1^{i''} + \Upsilon_{22}^i u_2^{i''} + \Upsilon_{23}^i u_3^{i''} + \Upsilon_{24}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_2^i \rangle \\ \Upsilon_{31}^i u_1^{i''} + \Upsilon_{32}^i u_2^{i''} + \Upsilon_{33}^i u_3^{i''} + \Upsilon_{34}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_3^i \rangle \\ \Upsilon_{41}^i u_1^{i''} + \Upsilon_{42}^i u_2^{i''} + \Upsilon_{43}^i u_3^{i''} + \Upsilon_{44}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_4^i \rangle \end{aligned} \quad (22)$$

Which gives us required derivatives. Thus from (20), we find the values of μ_i , which finally helps us to find the value of c_i in system (19). Thus the second curvature can be found from the second equation of (7), untill we find b_1 .

On using (7) we obtain $\kappa_1' = \langle \alpha''', n \rangle$.

3.3. Third curvature

To find the third curvature, we need to find the fourth derivative of the intersection curve α at P .

Since N_i is orthogonal to t , we may write $d_1N_1 + d_2N_2 + d_3N_3 + d_4N_4$ instead of $(-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2)n + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')b_1 + \kappa_1 \kappa_2 \kappa_3 b_2$ in $\alpha^{(4)}$, i.e.,

$$\alpha^{(4)} = -3\kappa_1 \kappa_1' t + d_1N_1 + d_2N_2 + d_3N_3 + d_4N_4 \quad (23)$$

Taking the dot product of (23) with N_1, N_2, N_3 and N_4 , we get

$$\begin{aligned}
\langle N_1, N_1 \rangle d_1 + \langle N_2, N_1 \rangle d_2 + \langle N_3, N_1 \rangle d_3 + \langle N_4, N_1 \rangle d_4 &= \xi_1 \\
\langle N_1, N_2 \rangle d_1 + \langle N_2, N_2 \rangle d_2 + \langle N_3, N_2 \rangle d_3 + \langle N_4, N_2 \rangle d_4 &= \xi_2 \\
\langle N_1, N_3 \rangle d_1 + \langle N_2, N_3 \rangle d_2 + \langle N_3, N_3 \rangle d_3 + \langle N_4, N_3 \rangle d_4 &= \xi_3 \\
\langle N_1, N_4 \rangle d_1 + \langle N_2, N_4 \rangle d_2 + \langle N_3, N_4 \rangle d_3 + \langle N_4, N_4 \rangle d_4 &= \xi_4
\end{aligned} \tag{24}$$

Where $\xi_i = \langle \alpha^{(4)}, N_i \rangle, i = 1, 2, 3, 4$ and $d_i \in \mathbb{R}$

Now to find d_i we have to find ξ_i . For that taking the dot product of $\alpha^{(4)}$ with N_i , we obtain

$$\begin{aligned}
\xi_r = & 4 \sum_{i,j=1}^4 \langle \Phi_{ij}^r, N_r \rangle u_i''' u_j' + 3 \sum_{i,j=1}^4 \langle \Phi_{ij}^r, N_r \rangle u_i'' u_j'' + 6 \sum_{i,j,k=1}^4 \langle \Phi_{ijk}^r, N_r \rangle u_i'' u_j' u_k' \\
& + \sum_{i,j,k,l=1}^4 \langle \Phi_{ijkl}^r, N_r \rangle u_i' u_j' u_k' u_l'
\end{aligned} \tag{25}$$

Since u_i', u_i'' are already known, so to find ξ_r we have to find u_i''' . These derivatives can be found by taking the product of both hand side of (10) with $\Phi_1^i, \Phi_2^i, \Phi_3^i$ and Φ_4^i , respectively. Hence, we can compute the Frenet vectors at P of the intersection curve by finding b_3, b_2 and b_1 - the third, second and first binormal vectors from the equations in (6) as now $\alpha', \alpha'', \alpha'''$ and $\alpha^{(4)}$ are at our disposal. Thus on using the binormal vector b_1 and (18), the second curvature of the intersection curve at P is obtained from the second equation of (7). Since κ_1, κ_2 and b_2 are already known, the third curvature can be now found from the third equation of (7).

3.4. Fourth curvature

To obtain the forth curvature κ_4 , we need to find the fifth derivative of the intersection curve of α at P . Similar to third and fourth derivative of the curve α , we may write

$$\alpha^{(5)} = \{-3(\kappa_1')^2 - 4\kappa_1 \kappa_2'' + \kappa_1^{(4)} + \kappa_1^2 \kappa_2^2\}t + m_1 N_1 + m_2 N_2 + m_3 N_4 + m_4 N_4 \tag{26}$$

Where, the system of equations for unknowns is

$$\begin{aligned}
\langle N_1, N_1 \rangle m_1 + \langle N_2, N_1 \rangle m_2 + \langle N_3, N_1 \rangle m_3 + \langle N_4, N_1 \rangle m_4 &= \eta_1 \\
\langle N_1, N_2 \rangle m_1 + \langle N_2, N_2 \rangle m_2 + \langle N_3, N_2 \rangle m_3 + \langle N_4, N_2 \rangle m_4 &= \eta_2 \\
\langle N_1, N_3 \rangle m_1 + \langle N_2, N_3 \rangle m_2 + \langle N_3, N_3 \rangle m_3 + \langle N_4, N_3 \rangle m_4 &= \eta_3 \\
\langle N_1, N_4 \rangle m_1 + \langle N_2, N_4 \rangle m_2 + \langle N_3, N_4 \rangle m_3 + \langle N_4, N_4 \rangle m_4 &= \eta_4
\end{aligned} \tag{27}$$

and $\eta_i = \langle \alpha^{(5)}, N_i \rangle$. Projecting (12) onto the unit vector N_i , respectively, we obtain η_i depending on $u_i^{(4)}$ besides u_i''', u_i'' and u_i' . Except $u_i^{(4)}$ all are known. So to find $u_i^{(4)}$ taking the scalar product of (11) with $\Phi_1^i, \Phi_2^i, \Phi_3^i, \Phi_4^i$, respectively. Consequently, the fourth curvature of the intersection curve can be found from the last equation of (7).

4. Darboux Frame, First Geodesic Curvature and First, Second and Third Geodesic Torsion.

In this section, we derive the Darboux frame $\{U_1^i, U_2^i, U_3^i, U_4^i, U_5^i\}$, the first geodesic curvature (κ_{1g}^i) and the first, second and third geodesic torsion (τ_{jg}^i), $j = 1, 2, 3$ for the transversal intersection curve of 4 parametric hypersurfaces in \mathbb{R}^5 .

Definition 4.1. Let M_i be a regular hypersurface in \mathbb{R}^5 and α be a curve on M_i . Then for each i , $1 \leq i \leq 4$, the function

$\kappa_{ig} : I \longrightarrow \mathbb{R}$ defined for $s \in I$ by

$$\kappa_{ig}(s) = \langle U_i'(s), U_{i+1}(s) \rangle$$

is called the **i^{th} geodesic curvature function** of the curve α and $\kappa_{ig}(s)$ is called the **i^{th} geodesic curvature** of the curve α at $\alpha(s)$.

4.0.1. Darboux Frame:

We are able to obtain a natural frame for the intersection curve-hypersurface pair $(\alpha(s), \mathbf{M}_i)$, i.e., the frame $\{U_1^i, U_2^i, U_3^i, U_4^i, U_5^i\}$ by using the Gram-Schmidt orthogonalization process. By assumption the sets $\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$ and $\alpha^{(4)}(s)$, and $\{N_1(p), N_2(p), N_3(p), N_4(p)\}$ are linearly independent.

Fixing $U_1^i = \alpha'(p)$ and $U_5^i = N_i$, we have

The natural frame (Darboux frame) $\{U_1^i, U_2^i, U_3^i, U_4^i, U_5^i\}$ is obtained, with $1 \leq i \leq 4$.

$$\begin{aligned} U_1^i &= \alpha'(p) \\ U_5^i &= N_i \\ U_2^i &= \frac{-\langle \alpha''(p), N_i \rangle N_i - \langle \alpha''(p), U_1^i \rangle U_1^i + \alpha''(p)}{\|-\langle \alpha''(p), N_i \rangle N_i - \langle \alpha''(p), U_1^i \rangle U_1^i + \alpha''(p)\|}, \\ U_3^i &= \frac{-\langle \alpha'''(p), N_i \rangle N_i - \langle \alpha'''(p), U_2^i \rangle U_2^i - \langle \alpha'''(p), U_1^i \rangle U_1^i + \alpha'''(p)}{\|-\langle \alpha'''(p), N_i \rangle N_i - \langle \alpha'''(p), U_2^i \rangle U_2^i - \langle \alpha'''(p), U_1^i \rangle U_1^i + \alpha'''(p)\|}, \\ U_4^i &= \frac{-\langle \alpha^{(4)}(p), N_i \rangle N_i - \langle \alpha^{(4)}(p), U_3^i \rangle U_3^i - \langle \alpha^{(4)}(p), U_2^i \rangle U_2^i - \langle \alpha^{(4)}(p), U_1^i \rangle U_1^i + \alpha^{(4)}(p)}{\|-\langle \alpha^{(4)}(p), N_i \rangle N_i - \langle \alpha^{(4)}(p), U_3^i \rangle U_3^i - \langle \alpha^{(4)}(p), U_2^i \rangle U_2^i - \langle \alpha^{(4)}(p), U_1^i \rangle U_1^i + \alpha^{(4)}(p)\|}. \end{aligned}$$

The j -th geodesic curvature κ_{jg}^i associated with i -th hypersurface is

$$\kappa_{jg}^i = \left\langle \left(U_j^i \right)', U_{j+1}^i \right\rangle = - \left\langle \left(U_{j+1}^i \right)', U_j^i \right\rangle, j \in \{1, 2, 3\}. \quad (28)$$

The j -th geodesic torsion τ_{jg}^i associated with i -th hypersurface is

$$\tau_{jg}^i = - \left\langle \left(U_5^i \right)', U_{j+1}^i \right\rangle = \left\langle \left(U_{j+1}^i \right)', U_5^i \right\rangle, j \in \{1, 2, 3\}. \quad (29)$$

4.0.2. First geodesic and j -th torsion geodesic Formulas

Theorem 4.1. First Geodesic Curvature and the j -th geodesic torsion of the intersection curve of 4 parametric hypersurfaces is

$$\kappa_{1g}^i = \sum_{j=1}^4 a_j \langle N_j, U_2^i \rangle \quad (30)$$

$$\tau_{jg}^i = - \left\langle \frac{(\bar{N}_i)_{u_1} u'_1 + (\bar{N}_i)_{u_2} u'_2 + (\bar{N}_i)_{u_3} u'_3 + (\bar{N}_i)_{u_4} u'_4}{\|\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i\|}, U_{j+1}^i \right\rangle, j = 1, 2, 3. \quad (31)$$

Proof.

For First Geodesic Curvature

$$\kappa_{1g}^i = \left\langle (U_1^i)', U_2^i \right\rangle = \left\langle \alpha''(s), U_2^i \right\rangle. \quad (32)$$

Now using (14), Eq. (30) follows easily.

For the j -th geodesic torsion, we need derivative of $U_5^i = N_i = \frac{\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i}{\|\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i\|}$. If defined

$N_i = \frac{\bar{N}_i}{\|\bar{N}_i\|}$, where $\bar{N}_i = \Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i$. Hence we derive

$$\frac{d}{ds}(U_5^i(s)) = \frac{dN_i}{ds} = \frac{\frac{d\bar{N}_i}{ds} \|\bar{N}_i\| - \bar{N}_i \frac{d(\|\bar{N}_i\|)}{ds}}{\|\bar{N}_i\|^2}$$

$$\tau_{jg}^i = - \left\langle \frac{dN_i}{ds}, U_{j+1}^i \right\rangle, j = 1, 2, 3.$$

$$\tau_{jg}^i = - \left\langle \frac{\frac{d\bar{N}_i}{ds} \|\bar{N}_i\|}{\|\bar{N}_i\|^2} - \frac{\bar{N}_i \frac{d(\|\bar{N}_i\|)}{ds}}{\|\bar{N}_i\|^2}, U_{j+1}^i \right\rangle, j = 1, 2, 3.$$

$$\tau_{jg}^i = - \left\langle \frac{\frac{d\bar{N}_i}{ds} \|\bar{N}_i\|}{\|\bar{N}_i\|^2}, U_{j+1}^i \right\rangle, j = 1, 2, 3., \text{ because } \left\langle \bar{N}_i, U_{j+1}^i \right\rangle = 0, j = 1, 2, 3.$$

$$\tau_{jg}^i = - \left\langle \frac{\frac{d\bar{N}_i}{ds}}{\|\bar{N}_i\|}, U_{j+1}^i \right\rangle, j = 1, 2, 3.$$

$$\frac{d\bar{N}_i}{ds} = (\bar{N}_i)_{u_1} u'_1 + (\bar{N}_i)_{u_2} u'_2 + (\bar{N}_i)_{u_3} u'_3 + (\bar{N}_i)_{u_4} u'_4$$

$$(\bar{N}_i)_{u_i} = \Phi_{1i}^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i + \Phi_1^i \times \Phi_{2i}^i \times \Phi_3^i \times \Phi_4^i + \Phi_1^i \times \Phi_2^i \times \Phi_{3i}^i \times \Phi_4^i + \Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_{4i}^i.$$

5. Example

Let M_1, M_2, M_3 and M_4 be the hypersurfaces given by, respectively

$$X^1(u_1, u_2, u_3, u_4) = (\sqrt{2} \sin u_3 \cos u_2 \cos u_1, \sqrt{2} \sin u_3 \sin u_2 \sin u_1, \sqrt{2} \sin u_3 \cos u_2, \sqrt{2} u_4 \cos u_3, \frac{1}{\sqrt{2}} \sin u_3)$$

$$X^2(u_1, u_2, u_3, u_4) = (u_3 \cos u_1 \cos u_2, u_3 \sin u_1 \cos u_2, u_3 \sin u_2, u_3 \cos u_2 \cos u_4)$$

$$X^3(u_1, u_2, u_3, u_4) = (u_3 \cos u_1 \cos u_2, u_3 \sin u_1 \cos u_2, u_4 \sin u_1, u_3 \sin u_1 \sin u_2)$$

$$X^4(u_1, u_2, u_3, u_4) = \left(\frac{1}{2} + \frac{1}{2} \cos u_1, \frac{1}{2} \sin u_1, u_2, u_3, \frac{u_4}{2} \right)$$

let us find the Frenet vectors and the curvatures of the intersection curve at the intersection point

$$p = X^1 \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, 1 \right) = X^2 \left(\frac{\pi}{4}, \frac{\pi}{4}, 1, \frac{\pi}{4} \right) = X^3 \left(\frac{\pi}{4}, \frac{\pi}{4}, 1, 1 \right) = X^4 \left(\frac{\pi}{2}, \frac{\sqrt{2}}{2}, 1, 1 \right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1, \frac{1}{2} \right)$$

The unit normals of these hypersurfaces are

$$N_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, 0, -\sqrt{\frac{2}{3}} \right), N_2 = \left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2}, \frac{1}{\sqrt{2}}, 0 \right) \\ , N_3 = \left(-\frac{2}{3}, 0, 0, \frac{1}{3}, -\frac{2}{3} \right), N_4 = (0, 1, 0, 0, 0)$$

The non-vanishing first fundamental coefficients are

$$g_{11}^1 = \frac{1}{2}, \quad g_{22}^1 = 1, \quad g_{33}^1 = \frac{9}{4}, \quad g_{44}^1 = 1, \quad g_{12}^1 = \frac{1}{2}, \quad g_{23}^1 = -\frac{1}{2}, \quad g_{34}^1 = -1 \\ g_{11}^2 = \frac{1}{2}, \quad g_{22}^2 = \frac{5}{4}, \quad g_{33}^2 = 2, \quad g_{44}^2 = \frac{1}{4}, \quad g_{24}^2 = \frac{1}{4}, \quad g_{11}^3 = \frac{5}{4}, \quad g_{22}^3 = \frac{3}{4}, \\ g_{33}^3 = \frac{3}{2}, \quad g_{44}^3 = \frac{1}{2}, \quad g_{12}^3 = \frac{1}{4}, \quad g_{14}^3 = \frac{1}{2}, \quad g_{23}^3 = -\frac{1}{2}, \quad g_{11}^4 = \frac{1}{4}, \quad g_{22}^4 = 1, \\ g_{33}^4 = 1, \quad g_{44}^4 = \frac{1}{4}$$

The unit tangent at the intersection point is found by

$$t = \frac{N_1 \otimes N_2 \otimes N_3 \otimes N_4}{\|N_1 \otimes N_2 \otimes N_3 \otimes N_4\|} = \left(-\frac{2}{\sqrt{91}}, 0, -5\sqrt{\frac{2}{91}}, -\frac{6}{\sqrt{91}}, -\frac{1}{\sqrt{91}} \right)$$

The non-vanishing second fundamental coefficients are

$$h_{11}^1 = -\frac{1}{\sqrt{6}}, \quad h_{22}^1 = -\frac{1}{\sqrt{6}}, \quad h_{12}^1 = -\frac{1}{\sqrt{6}}, \quad h_{11}^2 = \frac{1}{2\sqrt{2}}, \quad h_{22}^2 = \frac{1}{\sqrt{2}}, \quad h_{11}^3 = \frac{2}{3}, \quad h_{22}^3 = \frac{2}{3} \\ h_{12}^3 = -\frac{2}{3}, \quad h_{13}^3 = \frac{1}{3}, \quad h_{11}^4 = -\frac{1}{2}$$

From the linear system of equations in (17), we obtain

$$(u_1^1)' = -0.628971, \quad (u_2^1)' = 0.838628, \quad (u_3^1)' = -0.209657, \quad (u_4^1)' = -0.838628 \\ (u_1^2)' = 0.209657, \quad (u_2^2)' = -0.419314, \quad (u_3^2)' = -0.628971, \quad (u_4^2)' = 0.628971 \\ (u_1^3)' = 0.209657, \quad (u_2^3)' = -0.419314, \quad (u_3^3)' = -0.628971, \quad (u_4^3)' = -1.25794 \\ (u_1^4)' = 0.419314, \quad (u_2^4)' = -0.741249, \quad (u_3^4)' = -0.628971, \quad (u_4^4)' = -0.209651$$

Thus, we obtain $\kappa_n^1 = -0.879304$, $\kappa_n^2 = 0.139867$, $\kappa_n^3 = 0.117216$, $\kappa_n^4 = -0.0879121$. Hence, we have

$$\alpha'' = -1.321618N_1 - 0.469981N_2 + 0.698466N_3 - 0.285472N_4$$

Or,

$$\alpha'' = (-0.839029, -0.658857, 0.234991, -0.0995048, 0.613453)$$

Thus, $\kappa_1 = \|\alpha''\| = 1.25679$

Also the unit normal vector is

$$n = (-0.524421, -0.411807, 0.146877, -0.621938, 0.383428)$$

From (21), we get

$$\triangle_1 = (-1.05495, -1.14286, -0.279735, -0.395605, -0.021978),$$

$$\triangle_2 = (-0.32967, -0.417583, 0.248653, 0, -0.549451)$$

$$\triangle_3 = (-0.32967, -0.417583, -0.404061, 0, -0.197802), \triangle_4 = (0, -0.0879121, 0, 0, 0)$$

From the linear system of equations in (22), we obtain

$$\begin{aligned} (u_1^1)'' &= 0.106614, (u_2^1)'' = 0.161474, (u_3^1)'' = 0.89026, (u_4^1)'' = 1.05091 \\ (u_1^2)'' &= 0.268086, (u_2^2)'' = 0.366087, (u_3^2)'' = -0.309769, (u_4^2)'' = -2.69189 \\ (u_1^3)'' &= 0.454481, (u_2^3)'' = 0.795236, (u_3^3)'' = -0.141793, (u_4^3)'' = 0.450137 \\ (u_1^4)'' &= 1.67806, (u_2^4)'' = 0.2356, (u_3^4)'' = -0.234991, (u_4^4)'' = 1.22691 \end{aligned}$$

Which yields

$$\mu_1 = 0.4544, \mu_2 = -0.5105, \mu_3 = -1.4338, \mu_4 = -1.0554$$

Then, we have

$$\alpha''' = -(1.25679)^2 t + 1.84188N_1 + 0.451035N_2 - 2.14772N_3 - 1.64788N_4$$

$$\alpha''' = (2.35545, -1.0554, 0.945301, 0.596496, 0.0935034)$$

$$\kappa_1' = \langle \alpha''', n \rangle = -0.996915$$

Using α''' , we obtain

$$\begin{aligned} (u_1^1)''' &= -5.90731, (u_2^1)''' = 2.14186, (u_3^1)''' = 1.51393, (u_4^1)''' = 2.11042 \\ (u_1^2)''' &= -3.76546, (u_2^2)''' = 2.58181, (u_3^2)''' = 0.76716, (u_4^2)''' = -2.76882 \\ (u_1^3)''' &= -0.88196, (u_2^3)''' = -3.60164, (u_3^3)''' = -1.14366, (u_4^3)''' = 2.21882 \\ (u_1^4)''' &= -3.33833, (u_2^4)''' = 0.545801, (u_3^4)''' = 0.596496, (u_4^4)''' = 0.187007 \end{aligned}$$

Hence, we get

$$\xi_1 = -21.7826, \xi_2 = -3.3273, \xi_3 = 8.8342, \xi_4 = -1.3710$$

From (23), we have

$$\begin{aligned} \alpha^{(4)} &= -3(1.25679)(-0.996915) \left(-\frac{2}{\sqrt{91}}, 0, -5\sqrt{\frac{2}{91}}, -\frac{6}{\sqrt{91}}, -\frac{1}{\sqrt{91}} \right) - 41.7712N_1 \\ &\quad - 29.6654N_2 + 34.1873N_3 + 5.19372N_4 \end{aligned}$$

Or,

$$\alpha^{(4)} = (-30.1443, -1.371, 12.0465, -11.945, 10.9205)$$

Thus,

$$b_3 = \frac{\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}}{\|\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}\|} = (0.235522, 0.439366, -0.161734, -0.0297594, 0.851143),$$

$$b_2 = \frac{b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''}{\|b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''\|} = (0.0469859, -0.0544597, 0.64055, -0.763297, -0.0435799)$$

Also,

$$b_1 = \frac{b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''}{\|b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''\|} = (0.648319, -0.724303, -0.161293, -0.0530185, 0.16199)$$

Now curvature,

$$\kappa_2 = \frac{\langle \alpha''', b_1 \rangle}{\kappa_1} = 1.68888, \quad \kappa_3 = \frac{\langle \alpha^{(4)}, b_2 \rangle}{\kappa_1 \kappa_2} = 7.07463$$

Also from (4) we have, $\kappa_1'' = 35.3284$

Using $\alpha^{(4)}$, we obtain

$$\begin{aligned} (u_1^1)^{(4)} &= 44.1618, (u_2^1)^{(4)} = -21.5916, (u_3^1)^{(4)} = -4.64151, (u_4^1)^{(4)} = -16.5865 \\ (u_1^2)^{(4)} &= 22.5702, (u_2^2)^{(4)} = 33.779, (u_3^2)^{(4)} = -7.12615, (u_4^2)^{(4)} = 55.62 \\ (u_1^3)^{(4)} &= 26.8139, (u_2^3)^{(4)} = 22.2084, (u_3^3)^{(4)} = -4.81273, (u_4^3)^{(4)} = -9.77667 \\ (u_1^4)^{(4)} &= -0.6116, (u_2^4)^{(4)} = 16.2234, (u_3^4)^{(4)} = -11.945, (u_4^4)^{(4)} = 21.841 \end{aligned}$$

Hence, we have

$$\eta_1 = 194.9662, \quad \eta_2 = -27.5036, \quad \eta_3 = -56.7000, \quad \eta_4 = 29.6466$$

Thus we have

$$\begin{aligned} \alpha^{(5)} &= (-3(1.25679)^2 - 4(1.25679)(35.3284) + (1.25679)^4 + (1.25679)^2(1.68888)^2) t \\ &\quad + 313.53N_1 + 145.818N_2 - 210.771N_3 - 46.7969N_4 \end{aligned}$$

Or,

$$\alpha^{(5)} = (253.719, 29.6467, 57.0616, 143.136, -97.1016)$$

Thus, the fourth curvature is given by

$$\kappa_4 = \frac{\langle \alpha^{(5)}, b_3 \rangle}{\kappa_1 \kappa_2 \kappa_3} = -1.55521$$

Now, to find κ_{1g}^1 and τ_{jg}^1 , $j = 1, 2, 3$ for hypersurface M_1 , we have from (30) and (31), $\kappa_{1g}^1 = 0.584888$, $\tau_{1g}^1 = -0.774977$, $\tau_{2g}^1 = -0.0496875$, $\tau_{3g}^1 = -0.276372$. Similarly we can find κ_{1g}^i and τ_{jg}^i for M_2, M_3, M_4 .

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